Analytical Solution of Optimal Trajectory-Shaping Guidance

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This paper presents a combined midcourse and terminal guidance law design for missiles to achieve range enhancement with excellent intercept performance. We derive analytic solutions of a closed-loop, nonlinear optimal guidance law for three-dimensional flight for both the midcourse and terminal phases. This combined guidance law can quickly modify the missile trajectory during midcourse guidance when the target direction changes. Zero heading error is achieved at handover from the midcourse to the terminal phase. The guidance algorithm is in a feedback form, either in inertial coordinates or in seeker coordinates. It is sufficiently simple for onboard implementation and has been applied successfully for on-line operation.

Nomenclature

C_{D_0} = zero lift drag coefficient
C_{L_α} = lift coefficient curve slope $(\mathrm{d}C_L/\mathrm{d}\alpha)$
D = drag = $D_0 + D_L$
D_L = drag due to lift = $C_{D_{\alpha^2}}\alpha^2 qS = \eta C_{L_\alpha}\alpha^2 qS = \eta L_\alpha \alpha^2$
D_0 = zero lift drag = $C_{D_0} qS$
L = lift = $C_{L_\alpha}\alpha qS = L_\alpha \alpha$
L_α = lift curve slope $(\mathrm{d}L/\mathrm{d}\alpha) = C_{L_\alpha} qS$
m = mass
q = dynamic pressure = $\frac{1}{2}\rho V^2$
S = area
T = thrust
W = weight, mg
α = angle of attack
η = aerodynamic efficiency factor
ρ = atmospheric density

Introduction

A COMBINED missile midcourse and terminal guidance law design that maximizes the final velocity can lead to missile range enhancement. The guidance problem for range-enhanced missiles is to obtain an optimal trajectory and guidance law for midcourse and terminal phases. These missiles generally include command-update inertial midcourse guidance and active or semiactive radar seeker terminal guidance. The function of the midcourse guidance is to minimize energy loss and bring the heading error to zero at handover. Handover is the change from midcourse to terminal guidance and occurs following target acquisition by the missile seeker. In general, midcourse guidance is not fast and accurate enough, even at short range, to consistently achieve the desired miss distance that is within the lethal radius of the missile warhead; therefore a terminal guidance mode is necessary. Terminal guidance is the self-navigation phase following midcourse guidance during which the missile homes in on the target until intercept occurs and the missile warhead is detonated.

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The two types of midcourse guidance commonly used differ only in the coordinate systems employed. In one missile steering signals are directly supplied in an agreed-upon coordinate frame. In the other an inertial navigator is updated by externally supplied target information. The midcourse guidance law is usually a form of proportional navigation with appropriate trajectory-shaping modifications for minimizing energy loss.

The use of midcourse guidance followed by a relatively short period of terminal homing offers a significant improvement in fire power and missile intercept coverage at the expense of the inclusion of the sensors and data links necessary for implementation. The success or failure of the missile during the terminal guidance phase is affected by several factors. Seeker acquisition plays the primary role at the desired time of handover. The heading error at handover is an essential variable affecting the miss distance, depending on the missile-to-target range, missile speed, time of handover, autopilot and control loop dynamic response time lags, and guidance filter time constants. It is particularly important in the case of slower missile response and severe radome coupling as the speed and intercept altitude increase. As the missile approaches interception, speed and maneuverability become critical. Intercepting a crossing and/or maneuvering target is more difficult than intercepting a directly incoming, nonmaneuvering threat. Other factors affecting guidance accuracy and miss distance include target glint and severe fades in the target-reflected signal at a critical time prior to intercept.

Many long- and medium-range missile guidance studies have shown that optimal trajectory shaping promises an extended range with more favorable end-game conditions. For this optimal trajectory-shaping guidance law design, the combined guidance law for the midcourse and terminal phase is developed in this paper using optimal control techniques. However, in a three-dimensional, target-intercept flight, direct application of the optimal control theory will result in a two-point boundary value problem that involves several arbitrary parameters so that analytical solutions cannot be obtained without a lot of approximations. The problem can be further complicated with the lift, thrust and drag, and control constraints forced by structural and angle-of-attack limits. This increases the computation time so that it is not feasible to implement the resulting solutions of the missile performance within an onboard digital computer. In view of this complexity in the problem setup, either indirect methods[1] or direct methods based on nonlinear programming are used to solve the sensitivity and convergence problems. However, both methods require very fast onboard microprocessor technology for real-time, on-line operation.

In this paper a simpler, more easily implementable feedback guidance law for aerodynamic control is sought that would surpass the performance of these two methods. A practical approach has been used to obtain simplified, approximated closed-loop, nonlinear optimal guidance law that is applied to both the midcourse and terminal guidance phases of a three-dimensional, target-intercept flight. Its solutions are derived to satisfy the need for a real-time guidance law that can be easily implemented online within the computational capabilities of current microprocessor-based onboard systems. It has, in fact, been applied successfully with on-line operation and has led to an overall improvement in missile performance.

Although a surface-to-air missile model is used in the guidance law design, analysis, and simulation, the guidance law can also be applied to surface-to-surface, air-to-air, and air-to-surface type missiles.

Problem Formulation

The present position and velocity of the missile are expressed as

$$r(t_0)=(x_0,y_0,z_0) \tag{1}$$

$$V(t_0)=(\dot{x}_0,\dot{y}_0,\dot{z}_0) \tag{2}$$

where x is the longitudinal position, y the lateral position, and z the altitude; the dot represents a derivative with respect to time, and the subscript 0 represents the present time. The flight guidance problem of the missile is to find an acceleration command vector

$$a_c(t)=[a_{x_c}(t),\ a_{y_c}(t),\ a_{z_c}(t)] \tag{3}$$

for $t_0 \le t \le t_f$ such that at t_f, $r(t_f)$ and $V(t_f)$ would have reached some prespecified terminal condition with some cost function optimized. The subscript c denotes command and the subscript f denotes the final time.

First we derive simple guidance gains for the explicit guidance command equations in inertial coordinates. This technique is introduced in Ref. 2. Then we apply optimal control to find the analytical optimal guidance gains.

The steering equations with simple guidance gains are explicit functions of the current and the desired boundary conditions (position and velocity). Hence this guidance law is known as explicit guidance. Details of explicit guidance are provided in Ref. 2. Here we briefly summarize the derivation. The equations of motion with respect to a planet-centered, inertial Cartesian coordinate system are

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}=\boldsymbol{g}+\boldsymbol{a}_c \tag{4}$$

or

$$\ddot{x}=g_x+a_{x_c} \tag{5}$$

$$\ddot{y}=g_y+a_{y_c} \tag{6}$$

$$\ddot{z}=g_z+a_{z_c} \tag{7}$$

Integrating $\ddot{x}(t)$ from the present time t_0 to general time t will give

$$\dot{x}(t)-\dot{x}(t_0)=\int_{t_0}^{t}\ddot{x}(\tau)\mathrm{d}\tau \tag{8}$$

By integrating Eq. (8) from $t=t_0$ to $t=t_f$, we have

$$x(t_f)-x(t_0)-\dot{x}(t_0)t_g=\int_{t_0}^{t_f}\left[\int_{t_0}^{t}\ddot{x}(\tau)\mathrm{d}\tau\right]\mathrm{d}t \tag{9}$$

where time to go $t_g=t_f-t_0$. From Eq. (8) we have

$$\dot{x}(t_f)-\dot{x}(t_0)=\int_{t_0}^{t_f}\ddot{x}(\tau)\mathrm{d}\tau \tag{10}$$

The unknown function $\ddot{x}(t)$ can be expressed in a generalized Fourier series. However, Eqs. (9) and (10) can determine only two of these undetermined coefficients. In other words, there are only two coefficients C_1 and C_2 of $\ddot{x}(t)$ which satisfy Eqs. (9) and (10) such that

$$\ddot{x}(t)=C_1P_1(t)+C_2P_2(t) \tag{11}$$

where $P_1(t)$ and $P_2(t)$ are linearly independent, prespecified functions of time.

For implementation simplicity, we choose

$$P_1(t)=1 \qquad P_2(t)=t_g \tag{12}$$

Substituting Eqs. (11) and (12) into Eqs. (9) and (10), we obtain (see Ref. 2)

$$\ddot{x}(t)=(K_1/t_g)[\dot{x}(t_f)-\dot{x}(t_0)]+(K_2/t_g^2)\times[x(t_f)-x(t_0)-\dot{x}(t_0)t_g] \tag{13}$$

where $K_1=-2$ and $K_2=6$. Similarly, we can obtain

$$\ddot{y}(t)=(K_1/t_g)[\dot{y}(t_f)-\dot{y}(t_0)]+(K_2/t_g^2)\times[y(t_f)-y(t_0)-\dot{y}(t_0)t_g] \tag{14}$$

$$\ddot{z}(t)=(K_1/t_g)[\dot{z}(t_f)-\dot{z}(t_0)]+(K_2/t_g^2)\times[z(t_f)-z(t_0)-\dot{z}(t_0)t_g] \tag{15}$$

The functions chosen for $P_1(t)$ and $P_2(t)$ in Eq. (12) lead to particularly simple equations. This choice of functions is not necessarily the optimum choice, although it is probably the simplest and leads to quite a useful steering law. The gains $K_1=-2$ and $K_2=6$ satisfy only the boundary conditions [Eqs. (9) and (10)]. In order to satisfy the optimal trajectory of atmospheric flight, we want to find the optimal time-varying K_1 and K_2 that optimize a performance index and satisfy the constraints imposed on the solution so that a general acceleration vector can be expressed explicitly in a more general vector form as

$$\boldsymbol{a}=(K_1/t_g)(\boldsymbol{V}_f-\boldsymbol{V}_0)+(K_2/t_g^2)(\boldsymbol{r}_f-\boldsymbol{r}_0-\boldsymbol{V}_0t_g) \tag{16}$$

This guidance law is also an explicit function of the current and the desired boundary conditions, but with optimal time-varying gains to be derived later. From Eq. (4) the missile acceleration command in inertial coordinates $\boldsymbol{a}_c$ is equal to $\boldsymbol{a}$-$\boldsymbol{g}$ where $\boldsymbol{a}$ is from Eq. (16).

Figure 1 defines the nomenclature of a three-dimensional intercept geometry. The variables $\delta(R)$ and $\sigma(R)$ are functions of the slant range R (the dotted line connecting V and V_f). The variable δ is the predicted velocity angle error of the present and final vectors and σ is the heading error angle. The dotted triangle defines the plane containing V and R. To reduce the effect of uncontrolled axial acceleration command we let the first term of the guidance algorithm [Eq. (16)] at the normal direction be

$$+(K_1/t_g)V\sin\delta \tag{17}$$

The second term at the normal direction is reduced to

$$-(K_2/t_g)(V\sin\sigma/\cos\sigma) \tag{18}$$

Since $R \cong V\cos\sigma t_g$, the normal acceleration is written as

$$a = (K_1/R)V^2\sin\delta\cos\sigma - (K_2/R)V^2\sin\sigma \tag{19}$$

From Eq. (4) the missile acceleration command in seeker coordinates $\boldsymbol{a}_c$ is equal to $\boldsymbol{a}$-$\boldsymbol{g}$ where $\boldsymbol{a}$ is from Eq. (19).

The instantaneous curvature κ of the missile trajectory is defined as

$$\kappa = \frac{d\hat{\gamma}}{ds} = \frac{1}{V}\frac{d\hat{\gamma}}{dt} = \frac{1}{V^2}a \tag{20}$$

where $\hat{\gamma}$ is the flight angle about some inertial reference and $ds/dt = V$. From Eqs. (19) and (20), the optimal curvature κ is thus

$$\kappa = (K_1/R)\sin\delta\cos\sigma - (K_2/R)\sin\sigma \tag{21}$$

Analytic Optimal Guidance Law

The missile trajectory is generally defined in inertial coordinates with the acceleration [Eq. (16)] as the control normal to the missile velocity; however, it is more convenient to use the guidance algorithm in seeker coordinates [Eq. (19)], that is, a function of the feedback variables δ and σ, instead of the guidance algorithm in inertial coordinates [Eq. (16)], to derive the optimal guidance gains K_1 and K_2 with a performance index

$$J = \max V_f \tag{22}$$

and boundary conditions $\delta(t_f) = \sigma(t_f) = 0$. Maximizing the terminal speed in the performance index (22) for a maneuver from launch to the intercept point is equivalent to minimizing the total loss in the kinetic energy of the entire flight path and hence promising extended range with more favorable endgame conditions, which is the goal of this optimal trajectory shaping.

Equation (16) is a well defined guidance equation and is used as explicit guidance law. The gains $K_1 = -2$ and $K_2 = 6$ satisfy only the boundary conditions [Eqs. (9) and (10)] but not the optimality condition. Since it is difficult to directly derive the optimal gains in three-dimensional flight as stated in the Introduction, simplified and practical solutions are obtained from a reduced-order problem. We first derive the analytic optimal guidance gains K_1 and K_2 for the vertical plane guidance equations (13) and (15). Similar derivation is then done for the horizontal plane guidance equations (13) and (14). Then the gains derived from the vertical and horizontal plane guidance equations are applied to a three-dimensional flight using Eqs. (13-15) as guidance accelerations in inertial coordinates of X, Y, and Z directions, respectively. These equations are mutually independent and are three components of Eq. (16).

Vertical Plane Guidance

In the vertical plane we have the flight path angle γ equal to $\hat{\gamma}$ and

$$\delta(R) = \gamma_f - \gamma \tag{23}$$

$$\sigma(R) = \gamma + \theta \tag{24}$$

where θ is the inertial line-of-sight angle. The equations for R and θ are, respectively,

$$\frac{dR}{dt} = -V\cos\sigma \tag{25}$$

$$\frac{d\theta}{dt} = \frac{V\sin\sigma}{R} \tag{26}$$

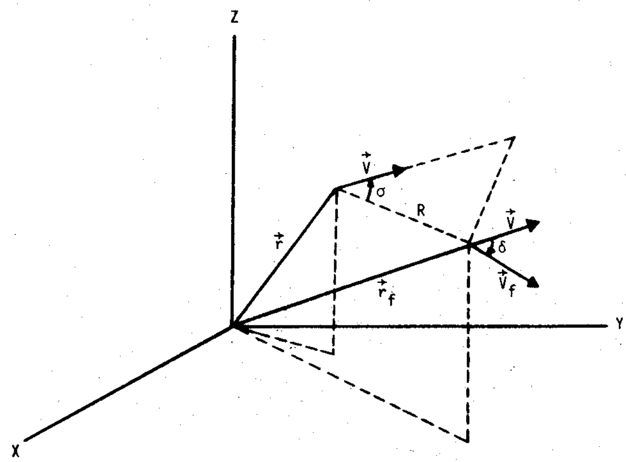

Fig. 1 Intercept geometry.

The state equations which govern the states γ and σ and the control κ are derived from Eqs. (20) and (24-26) as

$$\frac{d\gamma}{dR} = -\kappa\sec\sigma \tag{27}$$

and

$$\frac{d\sigma}{dR} = -\kappa\sec\sigma - \frac{1}{R}\tan\sigma \tag{28}$$

To obtain the optimum guidance law $a_c(t)$ we need to apply optimal control theory to find an optimal control law $\kappa(R)$ subject to the state Eqs. (27) and (28) and the boundary conditions $\sigma(t_f) = 0$, $\gamma(t_f) = \gamma_f$ (specified and updated in flight), and $R(t_f) = 0$ such that the missile terminal speed is maximized.

Maximizing the missile terminal speed is equivalent to maximizing

$$G = \int_R^0 \frac{dV}{dR} dR \tag{29}$$

We first find $G(t)$ as a simplified function of the state variables σ and γ, the control variable κ, and the missile characteristics. Then we apply optimal control theory to find the optimal κ such that $G(t)$ is maximized. The derivation of $G(t)$ is described in what follows.

The equations of motion, neglecting mass change due to fuel consumption, are

$$\frac{dV}{dt} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{30}$$

$$V\frac{d\gamma}{dt} = \frac{T\sin\alpha + L}{m} - g\cos\gamma \tag{31}$$

To obtain an analytic solution of optimal control κ, we simplify the derivation of $G(t)$ by assuming $\cos\alpha \simeq 1 - \alpha^2/2$ and using Eq. (30) so that

$$\frac{dV}{dt} = \frac{T}{m} - g\sin\gamma - \frac{D_0}{m}\left(1 + \frac{\eta L_\alpha + T/2}{D_0}\alpha^2\right) \tag{32}$$

and assuming $\sin\alpha \simeq \alpha$ and $L/W \gg \cos\gamma$ using Eqs. (20) and (31) to obtain

$$\alpha = \kappa m V^2 (T + L_\alpha)^{-1} \tag{33}$$

Then we substitute Eq. (33) into Eq. (32) and divide the result by Eq. (25) so that the range R is an independent variable.

Thus

$$\frac{\mathrm{d}V}{\mathrm{d}R}=\left[-\frac{T/m-g\sin\gamma}{V}+\frac{D_0}{mV}\left(1+\frac{\kappa^2}{2F^2}\right)\right]\sec\sigma \tag{34}$$

where the trajectory-shaping coefficient F is given by

$$F^2=\frac{D_0L_\alpha(T/L_\alpha+1)^2}{m^2V^4(2\eta+T/L_\alpha)} \tag{35}$$

and F in Eq. (35) depends on the missile characteristics. We then substitute Eq. (34) into Eq. (29) so that G becomes

$$G(t)=\frac{D_0}{mV}I-\int_R^0\frac{T/m-g\sin\gamma}{V}\sec\sigma\mathrm{d}R \tag{36}$$

where

$$I=\int_R^0\left(1+\frac{\kappa^2}{2F^2}\right)\sec\sigma\mathrm{d}R \tag{37}$$

From Eqs. (35) and (36) it is evident that trajectory shaping is influenced by variations in aerodynamic and propulsion parameters. The second term of Eq. (36) is equal to the sum of $\int_R^0(g\sin\gamma\ \sec\sigma/V\mathrm{d}R$ and $-\int_R^0(T/mV)\sec\sigma\mathrm{d}R$. The former can be deleted by neglecting the gravity term. For aerodynamic control, the latter can be approximated as a constant so that maximizing I is equivalent to maximizing $G(t)$, i.e., maximizing the missile terminal speed. Hence I becomes the new performance index.

The Hamiltonian to the variation problem is

$$H=H(\lambda_\sigma,\lambda_\gamma,\sigma,\gamma,\kappa,R)=\left(1+\frac{\kappa^2}{2F^2}\right)$$
$$\times\sec\sigma-\lambda_\sigma\frac{\tan\sigma}{R}-(\lambda_\sigma+\lambda_\gamma)\kappa\sec\sigma \tag{38}$$

where λ_i fulfills the adjoint equation

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}=-\frac{\partial H^*}{\partial x_i} \tag{39}$$

where H^* is the maximized Hamiltonian and x_i the state. Since the γ coordinate can be ignored, we have the integral

$$\lambda_\gamma=C_0 \tag{40}$$

The optimal variable κ is found to be

$$\kappa=F^2(\lambda_\sigma+C_0) \tag{41}$$

We can see that κ is proportional to $\lambda_\sigma+C_0$. From Eqs. (39-41), λ_σ for the unconstrained κ is

$$\frac{\mathrm{d}\lambda_\sigma}{\mathrm{d}R}=\frac{\sec^2\sigma}{F^2}\left[\left(\frac{\kappa^2}{2}-F^2\right)\sin\sigma+\frac{\kappa+C}{R}\right] \tag{42}$$

where $C=-C_0F^2$. We form an equation for κ by taking the derivative of Eq. (41). Then substituting Eq. (42) we obtain

$$\frac{\mathrm{d}\kappa}{\mathrm{d}R}=\left[\left(\frac{\kappa^2}{2}-F^2\right)\sin\sigma+\frac{\kappa+C}{R}\right]\sec^2\sigma \tag{43}$$

$$\frac{\mathrm{d}\kappa}{\mathrm{d}R}=\left[\left(\frac{\kappa^2}{2}-F^2\right)\sin\sigma+\frac{\kappa+C}{R}\right](1+\sin^2\sigma+\ldots) \tag{44}$$

By neglecting higher-order terms of $\sin\sigma$ and $\kappa^2\sin\sigma$ of Eq. (44), we have

$$R\frac{\mathrm{d}\kappa}{\mathrm{d}R}+F^2R\sin\sigma-\kappa-C=0 \tag{45}$$

From Eq. (28) we have

$$\frac{\mathrm{d}(R\sin\sigma)}{\mathrm{d}R}=-R\kappa \tag{46}$$

and substituting Eq. (46) into Eq. (45), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}R^2}(R\sin\sigma)-\frac{2}{R}\frac{\mathrm{d}(R\sin\sigma)}{\mathrm{d}R}-F^2R\sin\sigma+C=0 \tag{47}$$

We find that

$$R\sin\sigma=C/F^2+C_1e^{FR}(FR-1)+C_2e^{-FR}(FR+1) \tag{48}$$

satisfies Eq. (47). At the boundary condition $t=t_f$, we have

$$R_f=0 \qquad \sigma_f=0 \tag{49}$$

Using these boundary conditions in Eq. (48), we obtain

$$C=(C_1-C_2)F^2 \tag{50}$$

Then we substitute Eq. (50) into Eq. (48) to obtain

$$R\sin\sigma=C_1[e^{FR}(FR-1)+1]+C_2[e^{-FR}(FR+1)-1] \tag{51}$$

By differentiating Eq. (51) with respect to R and using Eq. (46) for comparison, we obtain

$$\kappa=-C_1F^2e^{FR}+C_2F^2e^{-FR} \tag{52}$$

We integrate Eq. (27) with κ from Eq. (52). In the integration, we use Eqs. (23) and (24) in the process and assume $(\gamma_f+\gamma)/2\simeq\gamma$ and $\cos\ [(\gamma_f-\gamma)/2]\simeq 1$ to obtain $\int_\gamma^{\gamma_f}\cos\sigma\mathrm{d}\gamma=\sin\delta\ \cos\sigma$. Thus we have

$$\sin\delta\cos\sigma/F=C_1(1-e^{FR})+C_2(1-e^{-FR}) \tag{53}$$

From Eqs. (51) and (53), we can obtain coefficients C_1 and C_2:

$$C_1=\frac{(1-e^{-FR})R\sin\sigma+[1-e^{-FR}(FR+1)]\sin\delta\cos\sigma/F}{e^{FR}(FR-2)-e^{-FR}(FR+2)+4} \tag{54}$$

$$C_2=\frac{(e^{FR}-1)R\sin\sigma+[e^{FR}(FR-1)+1]\sin\delta\cos\sigma/F}{e^{FR}(FR-2)-e^{-FR}(FR+2)+4} \tag{55}$$

Substituting C_1 and C_2 into Eq. (52), we have

$$\kappa=\{[F^2(2-e^{FR}-e^{-FR})]R\sin\sigma$$
$$+[2F^2R-F(e^{FR}-e^{-FR})]\sin\delta\cos\sigma\}$$
$$\div e^{FR}(FR-2)-e^{FR}(FR+2)+4 \tag{56}$$

Thus by equating Eqs. (21) and (56), the closed-loop feedback guidance gains in Eqs. (16) and (19) are analytically determined as

$$K_1=\frac{2F^2R^2-FR(e^{FR}-e^{-FR})}{e^{FR}(FR-2)-e^{-FR}(FR+2)+4} \tag{57}$$

$$K_2 = \frac{F^2R^2(e^{FR} + e^{-FR} - 2)}{e^{FR}(FR-2) - e^{-FR}(FR+2) + 4} \tag{58}$$

In the optimal guidance law, either in inertial coordinates [Eq. (16)] or in seeker coordinates [Eq. (19)], the first term considers the velocity error of the predicted terminal speed relative to the current velocity. The second term considers the position error of the predicted terminal position relative to the current position. The first term minimizes the energy loss, i.e., trajectory shaping, and the second term minimizes the time. Hence the combined weighted minimum-energy loss and minimum-time optimal intercept control trajectories are generated for a specified region of the predicted intercept point.

If the final flight path angle is not specified, then $\lambda_{\gamma_f} = 0$ and hence $C_0 = 0$ and $C = 0$. From Eq. (50) we obtain $C_1 = C_2$. Therefore from Eqs. (51) and (52) we obtain

$$\kappa_1 = \frac{-K_2}{R}\sin\sigma \tag{59}$$

where

$$K_2 = \frac{F^2R^2(e^{FR} - e^{-FR})}{e^{FR}(FR-1) + e^{-FR}(FR+1)} \tag{60}$$

and $K_1 = 0$.

Horizontal Plane Guidance

As in the case of the vertical plane guidance equation, we also derive the analytic optimal guidance gains K_1 and K_2 for the horizontal plane guidance equation. All the equations and optimal gain formulations in vertical flight are applied to the horizontal flight except that all gravity terms do not exist. In addition, α becomes β which is the sideslip angle, γ becomes ψ which is the heading angle, and θ is now the inertial line-of-sight angle in the horizontal plane. For ψ_f specified, K_1 and K_2 are the same as in Eqs. (57) and (58). For ψ_f not specified, $K_1 = 0$ and K_2 is the same as in Eq. (60). As noted earlier, the gravity term does not exist in the second term of $G(t)$ in Eq. (36), hence the assumption of this term as a constant is now more correct. The main concern is to minimize the lateral position error, i.e., the final time. Therefore ψ_f is not specified.

The preceding optimal guidance gains of the two planes are obtained by solving the guidance equations while satisfying the preceding constraints.

Real-Time Implementation and Performance

In general, real-time implementation is difficult in view of the two-point boundary value problem as well as the aerodynamic nonlinearities associated with the lift, thrust and drag, and the control constraints forced by structural and angle-of-attack limits. However, in our case we obtain analytic solutions that satisfy all optimality conditions and enforce a wide range of state and control constraints. The desired guidance acceleration commands are expressed explicitly in inertial coordinates by Eq. (16) in terms of the missile position and velocity at the present time and the predicted intercept point (PIP).

There are two methods of implementing this guidance law in real time. First, if the inertial reference unit (IRU) is available onboard, then it provides the missile its position and velocity. The target-tracking radar uplink provides the target position and velocity. The computing unit computes the time to go and the PIP by using the present missile acceleration, the missile-to-target relative velocity, and the missile-to-target range. Equation (16) is then used to compute the guidance acceleration commands in inertial coordinates. We then use the IRU quaternions to transform them into guidance commands in body coordinates and input to the autopilot. The second method of implementation is when there is no IRU onboard. Then Eq. (19) is used to compute the guidance acceleration command in seeker coordinates.

The control gains K_1 and K_2 are analytically derived and are nonlinear functions of the trajectory-shaping coefficient F and the range R. As noted previously, F depends on the missile aerodynamics and propulsion characteristics. The variables K_1 and K_2 are systematically updated in flight trajectory to reduce errors in the missile state; errors in the terminal state due to measurement error, target noise, and target maneuver; and autopilot and control loop dynamic response time lags. As pointed out previously, the heading error at handover is the main error affecting the miss distance. By radar specification, the noise distribution of the uplink data is known. Then covariance analysis is used to design the optimal guidance filter to attenuate noise effects (including measurement error of missile and target states) and hence reduce the heading error at handover. Therefore the analytical solution solves the boundary value problems of missiles intercepting targets in three-dimensional engagements. The resulting guidance law is computed in a feedback manner. The performance index and the optimal guidance law used, either in inertial coordinates or in seeker coordinates, contain equivalently a weighted combination of flight time and energy loss, and the intercept is regarded as a terminal constraint so that the optimal control will cause the missile to maneuver efficiently in terms of minimum energy loss and minimum time. A special case included is minimum time, as in the case of horizontal plane guidance Eq. (59), which uses the optimal gain K_2 from Eq. (60). Owing to the analytic form of the resulting feedback control law, the guidance algorithm is suitable for implementation in an airborne digital computer to demonstrate the feasibility of on-line optimal control operation.

Implementation and Simulation of Three-Dimensional Flight

In a three-dimensional engagement, the missile is guided during the midcourse portion of the flight by the optimal guidance law either in inertial coordinates or in seeker coordinates. For practicality, implementing this guidance law for three-dimensional flight in inertial coordinates is as follows.

In midcourse guidance we first compute three components of accelerations of the explicit guidance law (16). That is, we use K_1 and K_2 from Eqs. (57) and (58), respectively, to compute Eqs. (13) and (15) which are the X and Z direction accelerations, respectively, and use $K_1 = 0$ and K_2 from Eq. (60) to compute Eq. (14) which is the Y direction acceleration. Hence from Eq. (4) $\boldsymbol{a}_c$ is equal to $\boldsymbol{a}$-$\boldsymbol{g}$ where $\boldsymbol{a}$ has the X, Y, and Z acceleration components. We then transform $\boldsymbol{a}_c$ into guidance acceleration commands in body coordinates. For skid-to-turn (STT) missiles, Z and Y direction acceleration commands in body coordinates are used as commands to the autopilot. Hence the optimal guidance gains selected emphasize trajectory shaping and minimize energy loss and time in the vertical plane while minimizing mainly the lateral position error, i.e., time, in the horizontal plane.

When target maneuver is detected when the missile is approaching homing at the end of the midcourse phase, the horizontal plane guidance gain used remains the same; however, for the vertical plane guidance gains, trajectory shaping becomes less important while minimizing the position error for good accuracy in guidance is now more important. Hence we switch to either the same gain as in the horizontal plane guidance [Eq. (63)] or to $K_2 = K_1 + K_2$ and $K_1 = 0$, where K_1 and K_2 of $K_2 = K_1 + K_2$ are from Eqs. (57) and (58), respectively. Thus the vertical plane guidance equation in the terminal phase now minimizes the lateral position error, i.e., time. Therefore the time to go of the missile trajectory in homing is minimized for target interception. A shorter-range to go also results when we switch gains in the vertical plane equation when approaching handover as opposed to using the same guidance gains as in the midcourse phase all the way to intercept. This modification of optimal guidance gains at handover can achieve superior crossing-shot performance by turning the missile onto a favorable collision course early in the engagement and reducing heading error to zero at handover. It can quickly modify the missile trajectory during midcourse guidance when the target direction changes.

Thus the same guidance law, either in inertial coordinates (16) or in seeker coordinates (19), is used for both the midcourse and terminal phases. This modified optimal guidance law is easy to implement and works especially well for all-weather long-range missiles because it is simple and is either in inertial coordinates or in seeker coordinates. Implementation of this proposed guidance law in inertial coordinates is especially simple and useful and has been substantiated by real-life applications. Results show success in target interception if the initial heading error angle in the yaw plane at launch is within 70 deg.

Many realistic simulations using the proposed method have shown that very high performance is achieved in midcourse and terminal guidance. For surface-to-air missile applications the launch angle and the PIP range and altitude are used as scanning parameters to compute a family of three-dimensional intercept optimal trajectories that maximize the final speed at intercept and minimize both the intercept time and the missile's highest altitude. For real-time application the launch angle is a table look-up of the PIP range and altitude in a proportional relationship.

Comparison Between Optimal Trajectory and Nonoptimal Trajectory

With the optimal gains we can obtain smoother guidance acceleration command time histories than with the nonoptimal gains $K_1 = -2$ and $K_2 = 6$. For surface-to-air missile applications, the normal acceleration command history using the nonoptimal gains results in a higher altitude trajectory than that using the optimal gains.

The dynamic range and magnitude of the angle-of-attack and sideslip-angle time response using the optimal gains are much smaller than those using the nonoptimal gains. The angle-of-attack and sideslip-angle time response using the optimal gains are also smoother. Hence the optimal gains are suitable for air breathing missile applications where the dynamic range of the angle of attack and sideslip angle need to be constrained.

Since the optimal trajectory has smaller angle of attack and sideslip angle than the nonoptimal trajectory, it has much smaller heading error at handover. Therefore it is easier for the missile autopilot to control and follow the optimal trajectory with minimum control surface activity. Hence miss distance is improved with shorter flight time.

We approach the problem by assuming the second term of $G(t)$ in Eq. (36) as a constant and $L/W \gg \cos\gamma$ (i.e., neglecting the gravity term) in deriving the optimal gains to maximize the final speed at intercept. For missiles with larger $C_{D_\alpha 2}$, the optimal trajectory presents a significant improvement in the final speed over the nonoptimal trajectory since the drag contributed from the angle of attack is less than that from the nonoptimal trajectory. In some surface-to-air missile applications, the final speed in vertical flight has about a 15% improvement for a 50-km flight range. In horizontal flight the gravity term does not exist. The only assumption in deriving the optimal gains to maximize the final speed at intercept is the approximation of the second term of $G(t)$ in Eq. (36) as a constant. Therefore, implementing the proposed guidance law in a three-dimensional engagement has significant improvement (40%) in the final speed.

Conclusion

This paper presents completely closed-form solutions of optimal trajectory-shaping guidance that maximize the final speed at intercept. The proposed guidance law has been applied successfully to three-dimensional engagements. Since the guidance law formulation for the midcourse and terminal phases is the same, and can easily be implemented either in inertial coordinates or in seeker coordinates, it is very suitable for onboard application. In general, it is more appropriate for the missile to be guided during the midcourse portion of the flight by the optimal guidance law [Eq. (16)] in inertial coordinates. In terminal phase, the same guidance law, either in inertial coordinates [Eq. (16)] or in seeker coordinates [Eq. (19)], can be used. The major contribution of this guidance law is that it is very simple and useful, especially when applied in inertial coordinates. Application of this optimal trajectory shaping guidance law results in maximization of the final speed at intercept, improvement in miss distance, smoother acceleration command time history, zero heading error at handover, shorter flight time, smaller and smoother angle of attack and sideslip angle.

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